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An Improvement to the Fourier Series Method for Inversion of Laplace Transforms Applied to Elastic and Viscoelastic Waves

by Richard R. Laverty and George A. Gazonas

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A reprint from the *International Journal of Computational Methods*, vol. 3, no. 1, pp. 57–69, 2006.

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Richard R. Laverty
West Point, NY

George A. Gazonas
Weapons and Materials Research Directorate, ARL

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*Department of Mathematical Sciences, U.S. Military Academy, West Point, NY 10996

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14. ABSTRACT

A parametric study of composite strips leads to systems of partial differential equations, coupled through interface conditions, that are naturally solved in Laplace transform space. Because of the complexity of the solutions in transform space and the potential variations due to geometry and materials, a systematic approach to inversion is necessarily numerical. The Dubner-Abate-Crump (DAC) algorithm is the standard in such problems and is implemented. The presence of discontinuous wavefronts in the problems considered leads to Gibbs phenomenon; which, in turn, overestimates the values of maximum stress. These errors are mitigated by use of Lanczos' σ-factors, which combine naturally with the DAC algorithm.

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RICHARD R. LAVERTY

Department of Mathematical Sciences, United States Military Academy
West Point, New York 10996, USA
Richard.Laverty@usma.edu

GEORGE A. GAZONAS

Weapons and Materials Research Directorate, U.S. Army Research Laboratory
Aberdeen Proving Ground, Maryland 21005, USA
gazonas@arl.army.mil

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A parametric study of composite strips leads to systems of partial differential equations, coupled through interface conditions, that are naturally solved in Laplace transform space. Because of the complexity of the solutions in transform space and the potential variations due to geometry and materials, a systematic approach to inversion is necessarily numerical. The Dubner-Abate-Crump (DAC) algorithm is the standard in such problems and is implemented. The presence of discontinuous wavefronts in the problems considered leads to Gibbs phenomenon; which, in turn, overestimates the values of maximum stress. These errors are mitigated by use of Lanczos' σ-factors, which combine naturally with the DAC algorithm.

Keywords: Inverse laplace transform; Gibbs phenomenon; viscoelasticity; waves.

1. Introduction

The Laplace transform has proven to be the most natural method for solving the classical initial value problems of dynamic viscoelasticity. The transformation of a viscoelastic initial-boundary value problem (IBVP), by the correspondence principle, is an elastic boundary value problem (BVP), for which the solution is easily constructed. The difficulty then lies in the inversion of this transform. Although many methods exist for numerical inversion of Laplace transforms [Laverty (2003)], the Dubner-Abate-Crump (DAC) algorithm [Crump (1976); Dubner and Abate (1968); Durbin (1974)] has proven to be one of the simplest, yet most robust methods. Its implementation can be achieved in a computer program consisting of just a handful of lines. Its effectiveness can be measured by the frequency of its use, [Abate and Whitt (1995); Chen and Chou (1998); Frolov and Kitaev (1998); Georgiadis (1993); Georgiadis and Rigatios (1996); Georgiadis *et al.* (1999)].

Despite all its strengths, the DAC algorithm has one shortcoming within the context of wave propagation, Gibbs phenomenon, [Georgiadis *et al.* (1999); Laverty (2003)]. The algorithm itself is a construction of an approximate Fourier Series based upon Laplace transform data. Therefore, Gibbs phenomenon will be present at any discontinuity. In general, this will lead to over-estimation of the magnitude of a wavefront in the neighborhood of 10%. This is an unacceptable amount of error when considering optimization problems, such as those considered in [Velo and Gazonas (2003)], where the maximum stress of a two-layered elastic strip is optimized as a function of the impedance ratio of the two materials.

Our goal is to use the most natural analytic construction — the Laplace transform — to examine problems similar to Velo and Gazonas, but using viscoelastic strips. Since analytic inversion of the transforms is impractical for such a large class of problems, we will use the DAC algorithm. The problem of Gibbs phenomenon appears immediately when we attempt to verify the results for the elastic strips. In a review article by Gottlieb and Shu [1997], several methods are described for the mitigation of the Gibbs phenomenon, which were sorted into two classes; filter methods and expansion in orthogonal polynomials. We have found that the filter methods can be implemented very naturally with the DAC algorithm. Furthermore, the use of an expansion in a different basis has two drawbacks from our perspective. First, it requires the computation of the expansion coefficients in the new basis based on the Fourier expansion. Evaluation of these integrals must be done numerically and is a relatively high cost computation compared with the simplicity of the DAC algorithm. Second, the location of discontinuities needs to be known in advance to achieve rapid convergence of the new expansion. We are interested in scattering problems where tracking wavefronts (discontinuities) is not practical. Therefore, we have chosen to mitigate the Gibbs phenomenon via the filter methods; specifically, Lanczos' σ-factors, [Lanczos (1966); Gottlieb and Shu (1997)]. The adjustment to the standard DAC algorithm is easy to program and does not add appreciably to the computational burden of the inversion.

2. The Elastic/Elastic Strip

The following question was posed (and answered) by Velo and Gazonas [2003]: Can we find an impedance ratio for two perfectly bonded elastic strips such that the maximum stress propagated in each layer (strip) will be a minimum? The answer is yes. In fact, there exists an infinite sequence of discrete impedance ratios that satisfy this requirement. In this section, we attempt to verify this result via the DAC algorithm.

Consider the following coupled initial boundary value problems (IBVPs). The region $0 < x < L/2$ will be referred to as layer 1. A step in stress is applied to this layer at $x = 0$. The displacement in layer 1 is denoted by $u_1(x, t)$ and the elastic modulus and density are E and ρ, respectively. The partial differential equation (PDE) satisfied by $u(x,t)$ is

$$\rho\frac{\partial^2 u_1}{\partial t^2} = E\frac{\partial^2 u_1}{\partial x^2}, \qquad 0 < x < L/2, \;\; t > 0, \tag{1}$$

$$0 = u_1(x,0), \qquad 0 < x < L/2, \tag{2}$$

$$0 = \frac{\partial u_1}{\partial t}(x,0), \qquad 0 < x < L/2, \tag{3}$$

$$\Sigma_0 H(t) = -E\frac{\partial u_1}{\partial x}(0,t), \quad t > 0, \tag{4}$$

Layer 2 is the region $L/2 < x < L$. The displacement in layer 2 is denoted by $u_2(x,t)$, the elastic modulus and density are E/α and ρ/α, respectively. The PDE for $u_2(x,t)$ with the right end $(x = L)$ fixed is

$$\frac{\rho}{\alpha}\frac{\partial^2 u_2}{\partial t^2} = \frac{E}{\alpha}\frac{\partial^2 u_1}{\partial x^2}, \qquad L/2 < x < L, \;\; t > 0, \tag{5}$$

$$0 = u_2(x,0), \qquad L/2 < x < L, \tag{6}$$

$$0 = \frac{\partial u_2}{\partial t}(x,0), \qquad L/2 < x < L, \tag{7}$$

$$0 = u_2(L,t), \qquad t > 0. \tag{8}$$

To completely determine the solutions to Eqs. (1) through (8) we assume an ideal bonding at the interface: the displacement and stress are assumed continuous at $x = L/2$.

$$u_1(L/2-,t) = u_2(L/2+,t), \tag{9}$$

$$E\frac{\partial u_1}{\partial x}(L/2-,t) = \frac{E}{\alpha}\frac{\partial u_2}{\partial x}(L/2+,t). \tag{10}$$

With these materials the impedance ratio between layers is α and the wave speed, c, is the same in both layers.

$$c = \sqrt{E/\rho}. \tag{11}$$

We investigate how the impedance ratio will effect the maximum stress that will be propagated in each layer. To construct a solution we consider the associated BVPs and interface conditions in Laplace transform space

$$s^2\hat{u}_1(x;s) = c^2\hat{u}_1''(x;s), \quad 0 < x < L/2, \;\; \frac{\Sigma_0}{s} = -E\hat{u}_1'(0;s), \tag{12}$$

$$s^2\hat{u}_2(x;s) = c^2\hat{u}_2''(x;s), \quad L/2 < x < L, \;\; 0 = \hat{u}_2(L;s), \tag{13}$$

$$\hat{u}_1(L/2-;s) = \hat{u}_2(L/2+;s), \tag{14}$$

$$E\hat{u}_1'(L/2-;s) = \frac{E}{\alpha}\hat{u}_2'(L/2+;s), \tag{15}$$

where s is the transform variable, a prime denotes differentiation with respect to x and all transformed quantities are denoted by hats. The solution to Eqs. (12) through (15) can be constructed by elementary means, then the transform of the stress in each layer is given by Eqs. (16) and (17). The stresses will be inverted using the DAC algorithm.

$$E\hat{u}_1'(x;s) = \frac{\Sigma_0}{s}\left(\frac{1-\alpha}{2}\right)\left(\frac{\sinh(\frac{sL}{c})\sinh(\frac{sx}{c})}{\cosh^2(\frac{sL}{2c})+\alpha\sinh^2(\frac{sL}{2c})}\right) - \frac{\Sigma_0}{s}\cosh\left(\frac{sx}{c}\right), \quad (16)$$

$$\frac{E}{\alpha}\hat{u}_2'(x;s) = \frac{\Sigma_0}{s}\left(\frac{\cosh(\frac{s}{c}(x-L))}{\cosh^2(\frac{sL}{2c})+\alpha\sinh^2(\frac{sL}{2c})}\right). \quad (17)$$

Figure 1 shows the stress-time history for $\alpha = 2$ at the midpoint of each layer. The horizontal line inserted at a stress of $2\Sigma_0$ (relative stress of 2) is of special importance. It has been shown [Velo and Gazonas (2003)] that the stress in layer 1 will never exceed $2\Sigma_0$ and the maximum stress in layer 2 is bounded below by $2\Sigma_0$. This is verified in layer 2, where we see that the maximum stress is clearly greater than this value. However, there are times when the stress in layer 1 is beyond this limit. This is due to Gibbs phenomenon at the discontinuities in stress and is the drawback to the DAC algorithm that we wish to address in this paper.

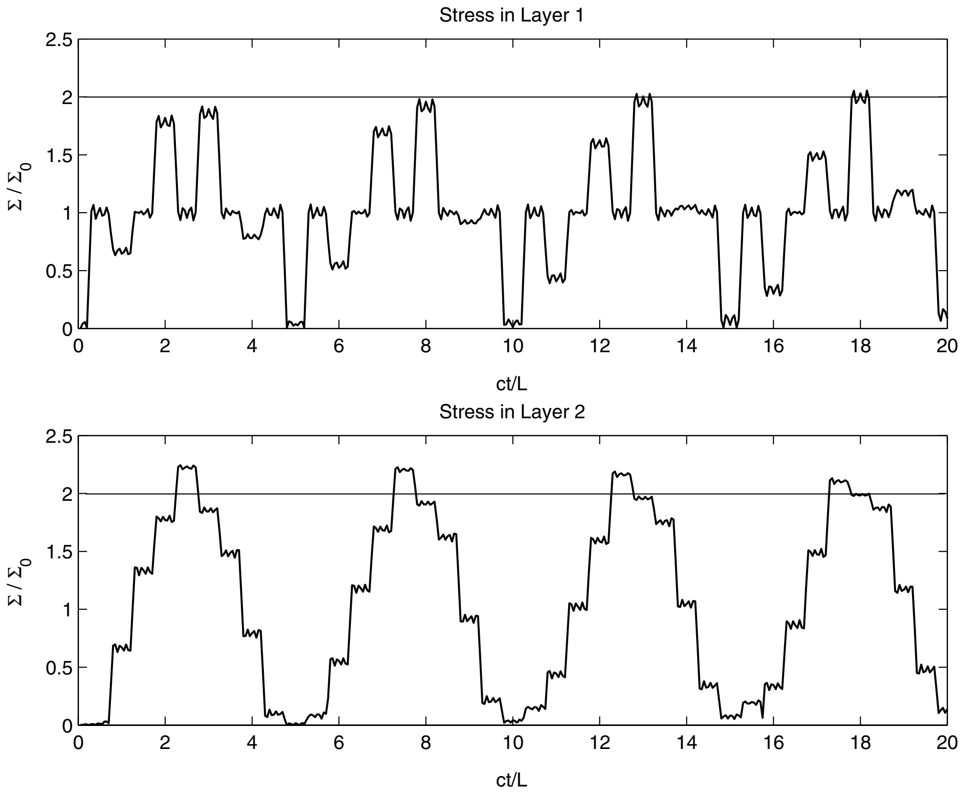

Fig. 1. The time history of stress at the layer midpoints for an impedance ratio of $\alpha = 2$ using the DAC algorithm with 256 terms and $tol = 10^{-3}$.

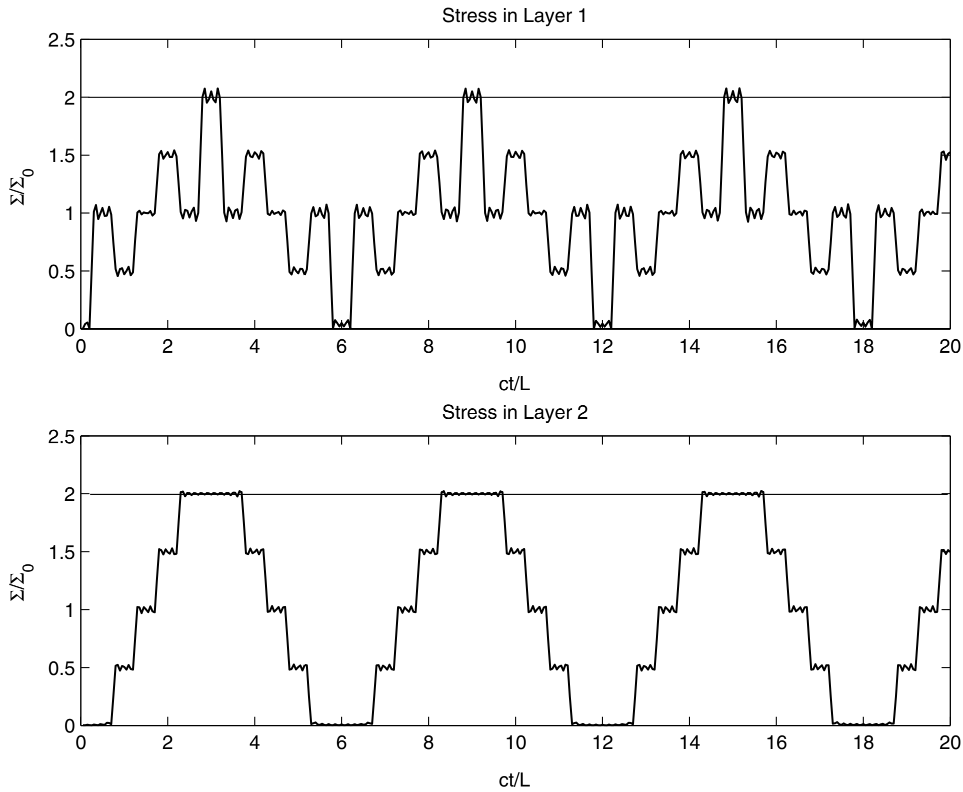

Fig. 2. The time history of stress at the layer midpoints for an impedance ratio of $\alpha = 3$ using the DAC algorithm with 256 terms and $tol = 10^{-3}$.

In Fig. 2 the impedance ratio has changed to $\alpha = 3$ and the Gibbs phenomenon in the layer 1 stress is more pronounced. Clearly, the DAC approximation is not faithful to the known bounds of maximum stress in the presence of discontinuous wavefronts.

Figure 3 is a graph of the maximum stress in each layer, as a function of α. The limit of $2\Sigma_0$ is clearly marked and we can see that as α grows there is no simple expression that can capture the values of the maximum stress. There is, however, clear values at which the maximum stress in layer 2 is at a minimum. It can be shown [Velo and Gazonas (2003)] that there exist an infinite number of discrete values of α for which the maximum stress in layer 2 will equal the limiting value of $2\Sigma_0$. Our values never reach all the way down to $2\Sigma_0$ (and our layer 1 values are seldom below $2\Sigma_0$) because of Gibbs phenomenon. Qualitatively, the shape of our graph agrees with that in Velo and Gazonas [2003]; and quantitatively, the critical values of α agree, but the DAC algorithm has not quantitatively captured the optimum values of maximum stress.

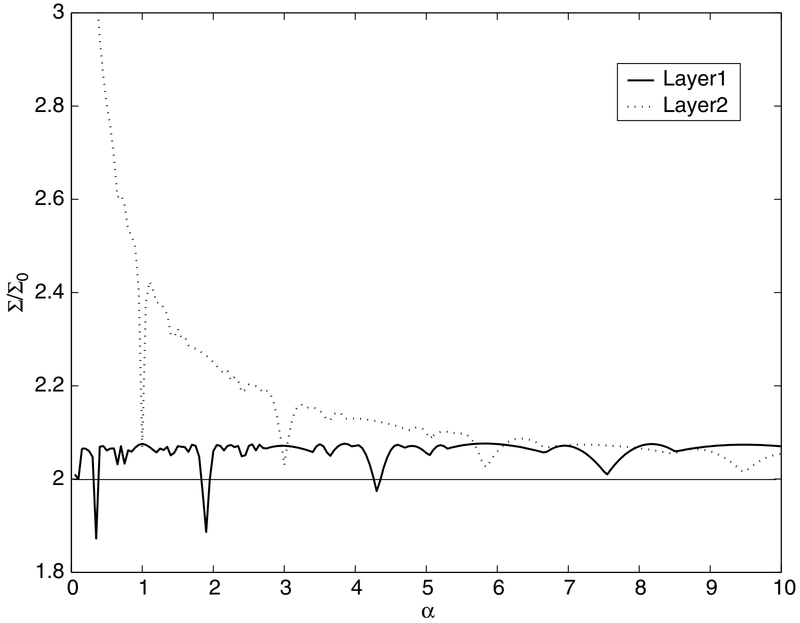

Fig. 3. The maximum stress at the layer midpoints as impedance ratio varies. Solutions were constructed using the DAC algorithm with 256 terms and $tol = 10^{-3}$.

3. Improvement of the DAC Algorithm

Given the Laplace transform $\hat{f}(s)$ we approximate the time domain function $f(t)$ with $\bar{f}(t)$.

$$\bar{f}(t) = \frac{e^{kt}}{T}\left[\frac{\hat{f}(a)}{2} + \sum_{n=1}^{\infty} \mathrm{Re}\left[\hat{f}\left(a + i\frac{n\pi}{T}\right)\right]\cos\left(\frac{n\pi}{T}t\right) - \sum_{n=1}^{\infty} \mathrm{Im}\left[\hat{f}\left(a + i\frac{n\pi}{T}\right)\right]\sin\left(\frac{n\pi}{T}t\right)\right]. \tag{18}$$

When we truncate the series Eq. (18), the function $\bar{f}(t)$ is the DAC approximation to $f(t)$.

Equation (18) is a Fourier Series for the function $\bar{f}(t)$ on the interval $(0, T)$. There are two parameters that we can control to achieve a desired accuracy; the truncation point N and the real number k. Obviously, as we increase N, we will increase the accuracy of our approximation. To achieve a given relative error we choose k according to

$$k = \xi - \frac{1}{2T}\ln(tol), \tag{19}$$

where *tol* is the bound on the relative error and ξ is a real number chosen slightly larger than the real part of the poles of $\hat{f}(s)$. When we know that $f(t)$ is bounded, we can choose $\xi = 0$.

Equation (18) can be derived directly from a trapezoid rule approximation of the exact inversion integral. However, determination of the parameter k can only be achieved through a more careful analysis based on the periodic extensions of $f(t)$ and the associated exact Fourier Series [Crump (1976); Dubner and Abate (1968); Durbin (1974)].

The Gibbs phenomenon, visible in Figs. 1 and 2, is the source of the errors in Fig. 3 that keep us from making accurate predictions of the smallest maximum stress propagated in layer 2 for a given value of α. However, methods do exist for mitigating the overshoot of Gibbs phenomenon, [Gottlieb and Shu (1997)]. For our purposes, the most effective approach is to implement a filter method. Although many filters exist and are all equally simple to include in our inversion algorithm we have chosen Lanzcos' σ-factors. Its performance makes it a good representative for the general class of filter methods.

To implement the σ-factors, we multiply each coefficient of the Fourier approximation Eq. (18) by a weight σ_n.

$$\sigma_n = \frac{\sin\left(\frac{n\pi}{N}\right)}{\frac{n\pi}{N}}, \tag{20}$$

where n is the series index and N is the index value at which we truncate the series. These σ-factors do not effect the convergence of the series, but they do smooth out the Gibbs phenomenon. Figure 4 gives the numerical inversion of Eqs. (16) and (17), using the same numerical parameters and impedance ratio as used in Fig. 2, but including the σ-factors.

In Figs. 2 and 4, we used one of the optimum values of the impedance ratio, $\alpha = 3$. We know that the stress should not cross $2\Sigma_0$ in either layer. It is clear that this fact is verified with our numerical inversion when we use the σ-factors (Fig. 4).

Now, we re-evaluate the maximum stress in each layer as α is varied. Figure 5 is a qualitative and quantitative, faithful reproduction of the analytical results obtained using the method-of-characteristics [Velo and Gazonas (2003)].

4. An Elastic/Viscoelastic Strip

With the confidence that the σ-factors provide us with a means to make accurate inversions, even in the presence of discontinuous wavefronts, we proceed to investigate a composite strip that is composed of an elastic and a viscoelastic material.

Consider a two layered composite occupying the region $0 < x < L$ where layer 1 is elastic and layer 2 is viscoelastic. We will place their interface at $x = l$. We maintain the notation that u_1 will be the displacement in layer 1 and u_2 is the displacement in layer 2. However, we must now introduce new notations for the density and stress. The density in layer 1 will be denoted by ρ; in layer 2 it will be

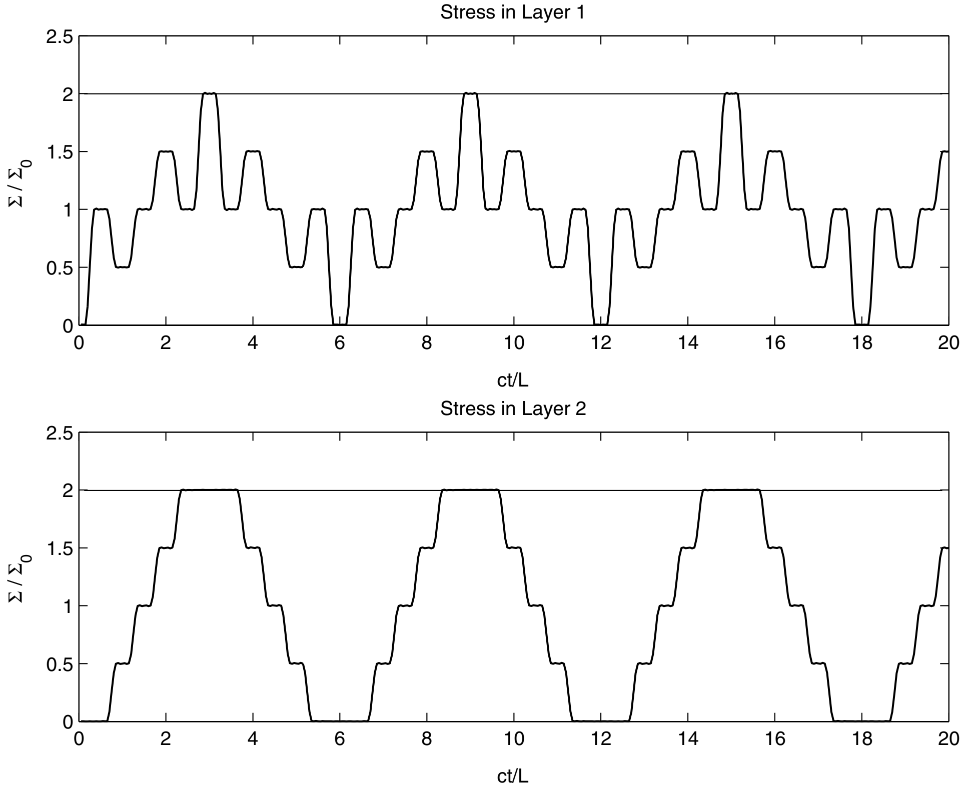

Fig. 4. The time history of stress at the layer midpoints for an impedance ratio of $\alpha = 3$ using the DAC algorithm with Lanczos' σ-factors, 256 terms and $tol = 10^{-3}$.

scaled by a factor of α. The stress functions will be $\Sigma_1(x,t)$ and $\Sigma_2(x,t)$ in layers 1 and 2, respectively. Using this notation, the IBVPs are:

$$\rho\frac{\partial^2 u_1}{\partial t^2} = E\frac{\partial^2 u_1}{\partial x^2}, \qquad 0 < x < l, \;\; t > 0, \tag{21}$$

$$0 = u_1(x,0), \qquad 0 < x < l, \tag{22}$$

$$0 = \frac{\partial u_1}{\partial t}(x,0), \qquad 0 < x < l, \tag{23}$$

$$\Sigma_0 H(t) = -E\frac{\partial u_1}{\partial x}(0,t), \quad t > 0. \tag{24}$$

$$\frac{\rho}{\alpha}\frac{\partial^2 u_2}{\partial t^2} = \frac{\partial \Sigma_2}{\partial x}, \qquad l < x < L, \;\; t > 0, \tag{25}$$

$$0 = u_2(x,0), \qquad l < x < L, \tag{26}$$

$$0 = \frac{\partial u_2}{\partial t}(x,0), \quad l < x < L, \tag{27}$$

$$0 = u_2(L,t), \qquad t > 0. \tag{28}$$

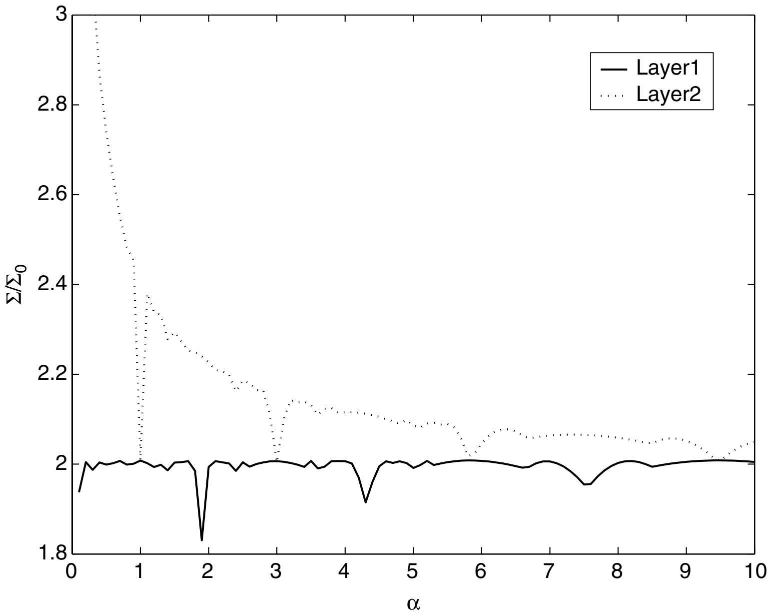

Fig. 5. The maximum stress of each layer (measured at the layer midpoint) as a function of impedance ratio, α, using Lanczos' σ factors. All approximations were made using $N = 256$ and $tol = 10^{-3}$.

We need an equation relating the stress τ and displacement w in layer 2. For a linear viscoelastic solid this can be accomplished using an hereditary integral

$$\Sigma_2(x,t) = \frac{\partial u_2}{\partial x}(x,t)G(0^+) + \int_{0^+}^{t} G'(t-\tau)\frac{\partial u_2}{\partial x}(x,\tau)d\tau, \tag{29}$$

where $G(t)$ is the relaxation modulus and prime denotes differentiation with respect to the argument. Equivalent forms of this constitutive law exist; for instance, after an integration by parts

$$\Sigma_2(x,t) = \int_{0^+}^{t} G(t-\tau)\frac{\partial^2 u_2}{\partial x \partial \tau}(x,\tau)d\tau. \tag{30}$$

However, the anticipated discontinuities in strain, w_x, make Eq. (29) more appropriate. There also exist differential forms of the constitutive law, but the convolution form of the hereditary integral formulation is convenient when the Laplace transform is applied.

To complete the model we add the interface conditions

$$u_1(l-,t) = u_2(l+,t), \tag{31}$$

$$\Sigma_1(l-,t) = \Sigma_2(l+,t). \tag{32}$$

Taking the Laplace transform of Eqs. (29), (32), and (21) through (28), we get the following BVPs and interface conditions for the transformed displacements.

$$s^2\hat{u}_1(x;s) = c^2\hat{u}_1''(x;s), \quad 0 < x < l, \; \frac{\Sigma_0}{s} = -E\hat{u}_1'(0;s), \tag{33}$$

$$s^2\hat{u}_2(x;s) = \hat{g}^2(s)\hat{u}_2''(x;s), \quad l < x < L, \; 0 = \hat{u}_2(L;s), \tag{34}$$

$$\hat{u}_1(l-;s) = \hat{u}_2(l+;s), \tag{35}$$

$$E\hat{u}'(l-;s) = \frac{\rho\hat{g}^2(s)}{\alpha}\hat{u}_2'(l+;s), \tag{36}$$

where c is the elastic wave speed, same as (11), $\hat{G}(s)$ is the Laplace transform of the relaxation modulus, and $\hat{g}(s)$ is given by

$$\hat{g}(s) = \sqrt{\frac{\alpha s\hat{G}(s)}{\rho}}. \tag{37}$$

The transformed stresses are

$$\hat{\Sigma}_1(x;s) = -\frac{\Sigma_0}{s}\left[\left(\frac{s^2\hat{G}(s)}{\hat{g}(s)}\right)\frac{\cosh\left(\frac{s}{\hat{g}(s)}(l-x)\right)\sinh\left(\frac{sx}{c}\right)}{\sinh\left(\frac{sl}{c}\right)d(l)} + \frac{\sinh\left(\frac{s}{c}(l-x)\right)}{\sinh\left(\frac{sl}{c}\right)}\right], \tag{38}$$

$$\hat{\Sigma}_2(x;s) = -\frac{\Sigma_0}{s}\left[\left(\frac{s^2\hat{G}(s)}{\hat{g}(s)}\right)\frac{\cosh\left(\frac{s}{\hat{g}(s)}(L-x)\right)}{d(l)}\right], \tag{39}$$

where

$$\begin{aligned} d(l) = {} & \frac{s^2\hat{G}(s)}{\hat{g}(s)}\cosh\left(\frac{sl}{c}\right)\cosh\left(\frac{s}{\hat{g}(s)}(L-l)\right) \\ & + \frac{sE}{c}\sinh\left(\frac{sl}{c}\right)\sinh\left(\frac{s}{\hat{g}(s)}(L-l)\right). \end{aligned} \tag{40}$$

Complicated expressions, such as Eqs. (38) through (40), were our original motivation to investigate numerical inversion techniques and have ultimately led us to the DAC algorithm. When we consider the daunting task of analytic inversion of these expressions, and then consider the variations in materials, characterized by $G(t)$, and configurations, more layers and varying widths, pragmatism demands numerical solution. Modification of the most appropriate technique, the DAC algorithm, is the purpose of this current study.

Figure 6 is the result of our modified DAC algorithm applied to Eqs. (38) through (40) using the relaxation modulus

$$G(t) = G_\infty + (G_0 - G_\infty)\, e^{-\beta t}. \tag{41}$$

Figure 6 also includes the solution for the same problem using the explicit finite element code DYNA3D [Whirley and Engelmann (1993)]. The parameter values are set to: $l = L/2$, $G_0 = E$, $G_\infty = 0.7E$ and $\beta = 1$. We can see in Fig. 6 that the two solution methods independently constructed identical solutions. We can also

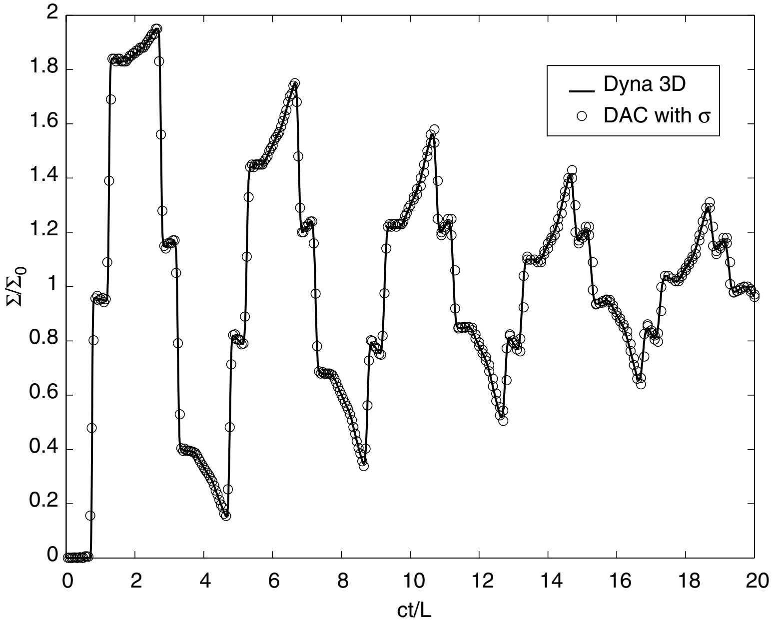

Fig. 6. Corroboration of the stress time history in the viscoelastic layer (measured at the layer midpoint) of an elastic/viscoelastic composite using the DAC algorithm (with σ smoothing) and DYNA3D. Layer 1 is elastic and layer 2 is a standard linear viscoelastic solid. The DAC approximations were made using $N = 256$ and $tol = 10^{-3}$.

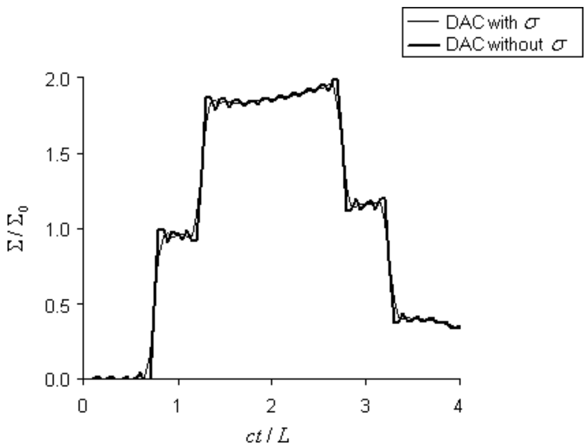

Fig. 7. Stress measured at the midpoint of the viscoelastic layer of the elastic/viscoelastic composite using the DAC algorithm with and without Lanczos' σ-factors.

see the effect of the viscoelastic solid is to "soften" the wavefronts and to dissipate the energy.

Figure 7 focuses on the first peak in Fig. 6. However, in Fig. 7 we have solved the same problem using only the DAC algorithm, with and without Lanczos' σ-factors. It is clear that the Gibbs phenomenon makes a noticable contribution to the peak stress, invalidating the measurement. Thus, the adapted algorithm we employ in this study is an essential part of any quantitative investigation of optimal designs.

5. Conclusion

Any parametric study of composite designs using viscoelastic materials will naturally lead to expressions such as Eqs. (38) and (39). The exact inversion of these transforms is an impractical task. After employing the most robust numerical method available, the DAC algorithm, we found that the Gibbs phenomenon corrupts our results to the extent that they do not match known, analytic solutions. Our search to mitigate these effects produced the Lanczos σ-factors: a general technique developed for use in Fourier series synthesis of functions and completely compatible with the DAC algorithm. The results of the DAC algorithm coupled with the σ-factors verified previous results and have provided means for further studies in viscoelastic composites.

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